

THE HOMOGENEOUS SPECTRUM OF MILNOR-WITT K -THEORY

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ABSTRACT. For any field F (of characteristic not equal to 2), we determine the Zariski spectrum of homogeneous prime ideals in $K_*^{MW}(F)$, the Milnor-Witt K -theory ring of F . As a corollary, we recover Lorenz and Leicht’s classical result on prime ideals in the Witt ring of F . Our computation can be seen as a first step in Balmer’s program for studying the tensor triangular geometry of the stable motivic homotopy category.

1. INTRODUCTION

In this note we completely determine the Zariski spectrum of homogeneous prime ideals in $K_*^{MW}(F)$, the Milnor-Witt K -theory of a field F . This graded ring contains information related to quadratic forms over F — in fact, $K_0^{MW}(F) \cong GW(F)$, the Grothendieck-Witt ring of F — and the Milnor K -theory of F , which appears as a natural quotient of $K_*^{MW}(F)$. While the prime ideals in $GW(F)$ are known classically via a theorem of Lorenz and Leicht [4] (see also [1, Remark 10.2]), we discover a more refined structure in $\mathrm{Spec}^h(K_*^{MW}(F))$, including a novel class of characteristic 2 primes indexed by the orderings on F which all collapse to the fundamental ideal $I \subseteq GW(F)$ in degree 0.

Much of the interest in $K_*^{MW}(F)$ stems from the distinguished role it plays in Voevodsky’s stable motivic homotopy category, $\mathrm{SH}^{\mathbb{A}^1}(F)$. Indeed, a theorem of Morel [7, §6, p 251] identifies $K_*^{MW}(F)$ with a graded ring of endomorphisms of the unit object in $\mathrm{SH}^{\mathbb{A}^1}(F)$. As $\mathrm{SH}^{\mathbb{A}^1}(F)$ is a tensor triangulated category (with tensor given by smash product, \wedge), it may be studied via Balmer’s methods of tensor triangular geometry [1]. More specifically, we can look at the full subcategory of compact objects, $\mathrm{SH}^{\mathbb{A}^1}(F)^c$. In this context, the goal is to determine the structure of the triangular spectrum $\mathrm{Spc}(\mathrm{SH}^{\mathbb{A}^1}(F)^c)$ of thick subcategories of $\mathrm{SH}^{\mathbb{A}^1}(F)^c$ which satisfy a “prime ideal” condition with respect to \wedge . Balmer’s primary tool in the study of triangular spectra is a naturally defined continuous map

$$\rho^\bullet : \mathrm{Spc}(\mathrm{SH}^{\mathbb{A}^1}(F)) \rightarrow \mathrm{Spec}^h(K_*^{MW}(F))$$

with codomain the Zariski spectrum of homogeneous prime ideals in $K_*^{MW}(F)$.

By identifying $\mathrm{Spec}^h(K_*^{MW}(F))$, we undertake a first step in Balmer’s program for studying the tensor triangular geometry of $\mathrm{SH}^{\mathbb{A}^1}(F)$. In particular, this raises

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the possibility of studying surjectivity properties of ρ^\bullet (which, in general, are unknown — see [1, Remark 10.5]) by explicitly constructing triangular primes lying over points in $\mathrm{Spec}^h(K_*^{MW}(F))$.

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Outline of the paper. The determination of $\mathrm{Spec}^h(K_*^{MW}(F))$ proceeds as follows. Section 2 gives general background on Milnor-Witt K -theory and states our main result. In subsections 3.1 and 3.2, the homogeneous spectra of two quotients of $K_*^{MW}(F)$ are determined, and in subsection 3.3 the two quotient spectra are stitched together to get the full spectrum.

2. MILNOR-WITT K -THEORY

The Milnor-Witt K -theory of a field, $K_*^{MW}(F)$, is a graded ring associated to a field by taking a certain quotient of the free algebra on a symbol η and the set of formally bracketed units in the field as follows:

Definition 2.1. For a set S , $[S] = \{[s] : s \in S\}$ is the set of (purely formal) symbols in S .

The free associative algebra on $[F^\times] \cup \{\eta\}$ is

$$\mathrm{Fral}([F^\times] \cup \{\eta\}) = \left\{ \sum_{1 \leq i \leq n} a_i \sigma_{i1}, \dots, \sigma_{ij_i} : a_i \in \mathbb{Z}, \sigma_{ij} \in [F^\times] \cup \{\eta\}, n \in \mathbb{N} \right\}$$

with multiplication and addition completely determined by the ring axioms.

Milnor-Witt K -theory, $K_*^{MW}(F)$, is the quotient $\mathrm{Fral}([F^\times] \cup \{\eta\})/K$, where K is the ideal generated by

- (1) $[ab] - [a] - [b] - \eta[a][b]$ (twisted logarithm),
- (2) $[a][b]$ where $a + b = 1$ (Steinberg relation),
- (3) $[a]\eta - \eta[a]$ (commutativity), and
- (4) $(2 + [-1]\eta)\eta$ (Witt relation).

We put a grading $K_*^{MW}(F)$ by declaring η to be of degree -1 and $[a]$ be of degree 1 for $a \in F^\times$.

We will usually suppress the field F in our notation. Note that $K_*^{MW}/(\eta)$ is simply Milnor K -theory. From this point on, we will assume that $\mathrm{char}(F) \neq 2$. This assumption is needed for Lemmas 2.3 and 3.1 and to make sure the orders defined in this paper are well-defined.

Morel [8] gives a presentation of K_*^{MW} as a pullback of Milnor K -theory and a ring L^* associated to the Witt ring. A correct proof of this presentation in every non-2 characteristic can be found in [3]. We define L^* and give Morel's pullback presentation below, though we will only make use of this presentation obliquely.

Definition 2.2. The ring L^* is defined by

$$L^* := \bigoplus_{n \in \mathbb{Z}} I^n \eta^{-n} \subseteq W(F)[\eta, \eta^{-1}]$$

where $W(F)$ is the Witt ring of F , I is the fundamental ideal in $W(F)$, and $I^n = W(F)$ for $n \leq 0$ by fiat.

Note that η has degree -1 in L^* , and $\eta^{-1} \notin L^*$. Note also the similarity between L^* and the Rees algebra of I in $W(F)$; indeed, L^* is essentially a \mathbb{Z} -graded version of the Rees algebra, sometimes called an extended Rees algebra. The ring L^* is elsewhere called I^* . We adopt this nonstandard notation to avoid confusion between L^1 and $I \subseteq L^0$.

The quotient of L^* by (I) is simply the graded ring associated with the I -adic filtration of the Witt ring,¹ $Gr_I = \bigoplus_{n \in \mathbb{N}} I^n / I^{n+1}$, which is, by the now settled Milnor conjecture [9, 6], isomorphic to $K_*^M / (2)$ via the Milnor map.

Lemma 2.3 (Morel, [8]; Gille-Scully-Zhong, [3]). *The Milnor-Witt K -theory ring, K_*^{MW} , is isomorphic to the pullback in the diagram below.*

$$\begin{array}{ccc} L^* \times_{Gr_I} K_*^M & \longrightarrow & K_*^M \\ \downarrow & & \downarrow q' \circ m \\ L^* & \xrightarrow{q} & Gr_I \end{array}$$

where q and q' are the quotients by (I) and (2) , respectively, and m is the Milnor map.

The ring K_*^{MW} is an ϵ -commutative graded ring,² with $\epsilon = -(1 + \eta[-1])$. We may then study its homogeneous spectrum, or the set of prime ideals generated by homogeneous elements:

$$\mathrm{Spec}^h(K_*^{MW}) = \{J \in \mathrm{Spec}(K_*^{MW}) : J = (J \cap K_n^{MW} : n \in \mathbb{Z})\}.$$

We equip $\mathrm{Spec}^h(K_*^{MW})$ with the Zariski topology generated by the subbasis elements

$$D(q) := \{J : q \notin J\}$$

as q ranges over K_*^{MW} . The main theorem of this paper is a complete characterization of $\mathrm{Spec}^h(K_*^{MW})$ in terms of the orderings³ on F .

Theorem 2.4. *Let $h = (2 + \eta[-1])$. As a set, the homogeneous spectrum is*

$$\mathrm{Spec}^h(K_*^{MW}) = A \amalg B \amalg C \amalg D \amalg \{([F^\times], 2), ([F^\times], \eta), ([F^\times], 2, \eta)\}$$

where

$$\begin{aligned} A &= \{([P_\alpha], h) : \alpha \text{ an ordering on } F\}, \\ B &= \{([P_\alpha], 2, \eta) : \alpha \text{ an ordering}\}, \\ C &= \{([P_\alpha], h, p) : \alpha \text{ an ordering, } p \text{ an odd prime}\}, \text{ and} \\ D &= \{([F^\times], \eta, p) : p \text{ an odd prime}\}. \end{aligned}$$

Topologically, $A \cong B \cong X_F$, where X_F is the space of orderings on F with the Harrison topology, that is the topology induced by the product topology on $\{\pm 1\}^{F^\times}$.

¹Note that (I) is the ideal generated by the copy of I in L^0 .

²Recall that a graded ring R_* is ϵ -commutative when $ab = \epsilon^{m+n}ba$ when $a \in R_m$ and $b \in R_n$.

³Recall that an ordering α on a field is uniquely determined by its positive cone P_α , and may be viewed as a group epimorphism $\alpha : F^\times \rightarrow \{\pm 1\}$ satisfying additivity: $\alpha(a+b) = 1$ if a, b are positive, i.e., when $\alpha(a) = \alpha(b) = 1$.

3. HOMOGENEOUS IDEALS OF K_*^{MW}

We determine the homogeneous spectrum of K_*^{MW} (as a topological space) in this section. From Morel's pullback presentation, $K_0^{MW} \cong W(F) \times_{Gr_I} \mathbb{Z} \cong GW(F)$, where $GW(F)$ is the Grothendieck-Witt ring of F . The image of the hyperbolic plane under this isomorphism is $h := (2 + \eta[-1])$. The Witt relation then says $h\eta = 0$. This suggests that computing $\text{Spec}^h(K_*^{MW})$ amounts to computing the ideals in $K_*^{MW}/(h)$ and $K_*^{MW}/(\eta)$. Indeed, this is literally the case if our interest is only in the spectra qua sets.⁴ We will start with these computations and then stitch the results together.

3.1. Determination of $\text{Spec}^h(K_*^{MW}/(h))$. We will need a more concrete presentation of $K_*^{MW}/(h)$. A lemma of Morel supplies one in terms of the ring L^* (described in §1).

Lemma 3.1 (Morel, [8]). *The quotient $K_*^{MW}/(h)$ is isomorphic to L^* via the map $[a] \mapsto \langle a, -1 \rangle \eta^{-1}$.*

As with the Witt ring (see [4]), the spectrum of L^* is deducible from the structure of the space, X_F , of orderings on F and the following computation of the spectrum for real closed fields.

Theorem 3.2. *If F is real closed, $\text{Spec}^h(L^*) = \{(p) : p \text{ an odd prime or } 0\} \amalg \{(\eta, 2), (2, L^1), (\eta, 2, L^1)\}$.*

Proof. In this case, $W(F) \cong \mathbb{Z}$, where the isomorphism carries I to $2\mathbb{Z}$. So, letting y be a generator for L^1 , $L^* = \mathbb{Z}[\eta, y]/(\eta y - 2)$. Let $J \in \text{Spec}^h(L^*)$, and let $p = \text{char}(J)$. (Recall that the characteristic of a prime ideal is the characteristic of its quotient ring.)

If $p = 0$, then we want to show that J contains no elements of degree 0, except 0. If any element of nonzero degree were in J , say ay^n or $a\eta^n$, then we would have $a2^n \in J$ in degree 0. So, $J = (0)$.

If p is an odd prime, then J descends to a homogeneous prime ideal in the integral domain $L^*/(p)$. Every homogeneous element of $L^*/(p)$ is invertible, meaning its only homogeneous prime ideal is (0) . So $J = (p)$.

If p is 2, then J contains either η or y or both. Clearly, $L^*/(2, \eta) \cong L^*/(2, y) \cong (\mathbb{Z}/2\mathbb{Z})[x]$, where x has degree -1 or 1 , respectively. The only homogeneous prime ideals here are (x) and (0) , which pull back to $(2, \eta, y)$ and one of $(2, \eta)$ or $(2, y)$, respectively. \square

Since the ring will show up again, we will write

$$R := \mathbb{Z}[\eta, y]/(\eta y - 2).$$

We will also make use of the following definitions.

Definition 3.3. For any fields F, K , and any ring morphism $\phi : W(F) \rightarrow W(K)$, the canonical homogeneous extension $\phi^+ : L^*F \rightarrow L^*K$ is given by:

$$\phi^+(q\eta^n) = \phi(q)\eta^n$$

⁴Secretly, each of these quotients is one of the factors in the pullback presentation of §1. The spectrum of K_*^{MW} is then the pushout of the spectra of the quotients. We will do a little extra computation to avoid proving the general result about ϵ -commutative graded rings. See [11] for the commutative non-graded case.

We will want some notation for the the specific case of the signature maps.⁵ For $\alpha \in X_F$, let F_α be the real closure of F at α and consider $\text{sgn}_\alpha : W(F) \rightarrow W(F_\alpha) \cong \mathbb{Z}$ for $\alpha \in X_F$. We will write J_α^+ for $\ker(\text{sgn}_\alpha^+)$.

The next two theorems completely list the elements of $\text{Spec}^h(L^*)$.

Theorem 3.4. *The following are homogeneous prime ideals in L^* :*

$$J_\alpha^+, (J_\alpha^+, p), (J_\alpha^+, 2, \eta), (L^1), (L^1, \eta) \text{ for } \alpha \in X_F \text{ and } p \text{ an odd prime.}$$

Furthermore, any homogeneous ideal containing some J_α^+ is one of these, and if $\alpha \neq \beta$, $J_\alpha^+ \neq J_\beta^+$, $(J_\alpha^+, p) \neq (J_\beta^+, p)$, and $(J_\alpha^+, 2, \eta) \neq (J_\beta^+, 2, \eta)$.

Proof. We know that $L^*/J_\alpha^+ \cong R$, and R has homogeneous spectrum

$$\{(p) : p \text{ an odd prime or } 0\} \amalg \{(\eta, 2), (2, L^1), (\eta, 2, L^1)\}.$$

The preimage of a spectrum under a quotient map can be computed in the ϵ -graded commutative in the same way as in the commutative case (cf. [10, Vol. 2. Ch VII, §2, Lemma 1] for the graded case). Clearly, (0) , (p) , $(\eta, 2)$ pull back to J_α^+ , (J_α^+, p) , and $(J_\alpha^+, 2, \eta)$ respectively. Also, since $\alpha(a) = \pm 1$, $\sum_{i=1}^n \alpha(a_i) = 0$ implies n must be even. so $J_\alpha^+ \subseteq (I) = (L^1)$. Thus $(2, L^1)$ and $(2, \eta, L^1)$ pull back to (L^1) and (L^1, η) . So, $\text{Spec}^h(R)$ pulls back to the ideals listed.

If $\beta \neq \alpha$, then there is some a such that $\text{sgn}_\alpha^+(\langle 1, a \rangle) = 2\eta^{-1}$ and $\text{sgn}_\beta^+(\langle 1, a \rangle) = 0$ (namely, any element of F^\times on which α and β disagree), and $2\eta^{-1}$ is not in any of J_α^+ , (J_α^+, p) , or $(J_\alpha^+, 2, \eta)$ as otherwise each of these would contain every element of positive degree in L^* . \square

The next theorem implies there are no other ideals than the ones listed in Theorem 3.4. There is a much simpler proof for the non-two characteristic part of the spectrum, but we will present only the most general proof below.

Theorem 3.5. *Every element of $\text{Spec}^h(L^*)$ contains some J_α^+ .*

Proof. Suppose $J \in \text{Spec}^h(L^*)$. If $L^1 \subseteq J$, then for any $\alpha \in X_F$, again since $J_\alpha^+ \subseteq (L^1)$, $J_\alpha^+ \subseteq J$. So, we may assume $L^1 \not\subseteq J$.

For every $a \in F^\times$, in L^* ,

$$(\langle 1, a \rangle \eta^{-1})(\langle 1, -a \rangle \eta^{-1}) = 0$$

so

$$\langle 1, a \rangle \eta^{-1} \in J, \text{ or } \langle 1, -a \rangle \eta^{-1} \in J$$

Thus, $\langle a, b \rangle \eta^{-1} \equiv \langle \epsilon_0, \epsilon_1 \rangle \eta^{-1}$ where $\epsilon_i \in \{\pm 1\}$. So, $\langle a, b \rangle \eta^{-1} = \pm 2\eta^{-1}$ or 0 modulo J .

Note that, since $L^1 \not\subseteq J$ and L^1 is generated by elements of the form $\langle a, b \rangle \eta^{-1}$, $2\eta^{-1} \notin J$. It follows that $\langle -a, \alpha(a) \rangle \eta^{-1} \in J$ characterizes a unique function $\alpha \in \{\pm 1\}^{F^\times}$. We claim that α is an order and $J_\alpha^+ \subseteq J$.

For the first claim, clearly $\alpha(1) = 1$. Let a, b be such that $\alpha(a) = \alpha(b) = 1$. It suffices to show that $\alpha(-a) = -1$ and $\alpha(ab) = \alpha(a+b) = 1$. First note:

$$\langle -1, -(-a) \rangle \eta^{-1} = -\langle 1, -a \rangle \in J$$

so $\alpha(-a) = -1$. Consider the product:

$$0 \equiv \langle 1, -a \rangle \langle 1, -b \rangle \eta^{-1} \equiv \langle 1, -a, -b, ab \rangle \eta^{-1} \equiv \langle -b, ab \rangle \eta^{-1} \equiv \langle -1, ab \rangle \eta^{-1} \pmod{J}$$

⁵Recall that the signature of a form $q = \langle a_1, \dots, a_n \rangle$ at the order α is $\text{sgn}_\alpha(q) = \sum_{1 \leq i \leq n} \alpha(a_i)$.

so $\alpha(ab) = 1$. We then have, working modulo J and using the classical identity $\langle a, b \rangle \cong \langle a + b, ab(a + b) \rangle$:

$$\begin{aligned}
(2\eta^{-1})(\langle 1, a + b \rangle \eta^{-1}) &\equiv (\langle 1, 1 \rangle \eta^{-1})(\langle 1, a + b \rangle \eta^{-1}) \\
&\equiv (\langle 1, ab \rangle \eta^{-1})(\langle 1, a + b \rangle \eta^{-1}) && (\text{since } \alpha(ab) = 1) \\
&\equiv \langle 1, ab, (a + b), (a + b)ab \rangle \eta^{-2} \\
&\equiv \langle 1, ab, a, b \rangle \eta^{-2} && (\text{by the identity}) \\
&\equiv (2\eta^{-1})(2\eta^{-1}).
\end{aligned}$$

Since $2\eta^{-1} \notin J$, we get $\langle 1, a + b \rangle \eta^{-1} \notin J$. Thus $\alpha(a + b) = 1$.

For the second claim, note that for all $a_i \in F^\times$,

$$\text{sgn}_\alpha^+ \left(\left(\sum \langle a_i \rangle \right) \eta^n \right) = \left(\sum \langle \alpha(a_i) \rangle \right) \eta^n \equiv \left(\sum \langle a_i \rangle \right) \eta^n \pmod{J}.$$

□

It is straightforward to move the homogeneous spectrum of L^* through the isomorphism of Lemma 3.1 ($[a] \mapsto \langle a, -1 \rangle \eta^{-1}$) to get the spectrum of $K_*^{MW}/(h)$.

Corollary 3.6. *If $J \in \text{Spec}^h(K_*^{MW}/(h))$, J is exactly one of the following:*

- $([P_\alpha])$ for some $\alpha \in X_F$,
- $([P_\alpha], p)$ for some $\alpha \in X_F$, odd prime p ,
- $([P_\alpha], 2, \eta)$ for some $\alpha \in X_F$,
- $([F^\times])$, or
- $([F^\times], \eta)$.

To determine the topological structure of the homogeneous spectrum of L^* (with the topology induced by the Zariski topology), we will need a small lemma:

Lemma 3.7. *If R is a graded ring, the sets $D(q) = \{J \in \text{Spec}(R) : q \notin J\}$ restricted to homogeneous q form a subbasis for the Zariski topology on $\text{Spec}^h(R)$.*

Proof. Consider $q = q_1 + \cdots + q_n \in R$, where the q_i are homogeneous. Then,

$$\begin{aligned}
D(q) &= \{J \in \text{Spec}^h(R) : q \notin J\} \\
&= \{J \in \text{Spec}^h(R) : \exists i (q_i \notin J)\} \\
&= \bigcup_{1 \leq i \leq n} \{J \in \text{Spec}^h(R) : q_i \notin J\} \\
&= \bigcup_{1 \leq i \leq n} D(q_i).
\end{aligned}$$

So the $D(q_i)$ generate the same topology as the $D(q)$. □

Much of the topological information about $\text{Spec}^h(L^*)$ is coded by the topological structure of its minimal ideals. Let MinSpec^h denote the subspace of these minimal ideals.

Theorem 3.8. *The minimum homogeneous spectrum $\text{MinSpec}^h(L^*)$ is homeomorphic to X_F with the Harrison topology.*

Proof. From previous computation, $\text{MinSpec}^h(L^*) = \{J_+^\alpha : \alpha \in X_f\}$. We will show that the obvious bijection σ_0 (given by $\alpha \mapsto J_+^\alpha$) is a homeomorphism.

To see that σ_0 is continuous, consider the subbasic open set $D(q)$ in $\text{MinSpec}^h(L^*)$, where q is homogeneous. We have that $q = \tilde{q}\eta^n$ for some $\tilde{q} \in W(F)$.

$$\begin{aligned}\sigma_0^{-1}(D(q)) &= \{\alpha : \text{sgn}_\alpha^+(q) \neq 0\} \\ &= \{\alpha : \text{sgn}_\alpha(\tilde{q}) \neq 0\} \\ &= \text{sgn}(\tilde{q})^{-1}(\mathbb{Z} \setminus \{0\}),\end{aligned}$$

where $\text{sgn}(\tilde{q}) : X_F \rightarrow \mathbb{Z}$ is the total signature given by $\text{sgn}(\tilde{q})(\alpha) = \alpha(\tilde{q})$. Note that if $\tilde{q} = \langle a_1, \dots, a_n \rangle$, then $\text{sgn}(\tilde{q}) = \sum_{1 \leq i \leq n} \text{sgn}(\langle a_i \rangle)$. Giving \mathbb{Z} the discrete topology, each $\text{sgn}(\langle a_i \rangle)$ is continuous by the definition of the Harrison topology, so $\text{sgn}(\tilde{q})$ is a sum of continuous functions and thus continuous. So, the set $\text{sgn}(\tilde{q})^{-1}(\mathbb{Z} \setminus \{0\})$ is open in X_F .

To see that σ_0 is open, consider the subbasic open set $H(a) = \{\alpha : \alpha(a) = 1\}$ in X_F :

$$\begin{aligned}\sigma_0(H(a)) &= \{J_+^\alpha : \alpha(a) = 1\} \\ &= \{J_+^\alpha : \text{sgn}_\alpha^+(\langle 1, a \rangle) \neq 0\} \\ &= D(\langle 1, a \rangle),\end{aligned}$$

which is open. \square

A fuller description of the topology is subsumed by the description of the topology on $\text{Spec}^h(K_*^{MW})$.

3.2. Determination of $\text{Spec}^h(K_*^{MW}/(\eta))$. Again, note that $K_*^{MW}/(\eta)$ is isomorphic to K_*^M . The relevant arithmetical facts about this ring are established in Milnor's original paper [5]. As with the quotient by (h) , the spectrum of the quotient by (η) is described in terms of orderings. Efrat studies the relation between orderings and quotients of K_*^M in [2], and his results are closely related to this computation.

We can immediately reduce the computation of the spectrum to the case where the ideals are of characteristic 2.

Lemma 3.9.

$$\text{Spec}^h(K_*^M) \setminus \{J : 2 \in J\} = \{(K_1^M, p) : p \text{ an odd prime or } 0\}$$

Proof. Suppose $J \in \text{Spec}^h(K_*^M)$ and $\text{char}(J) \neq 2$. Then, since $2[-1] = 0$, $[-1] \in J$. We then have, for all $a \in F^\times$

$$[a, a] \equiv [a, -1] \equiv 0 \pmod{J}.$$

So $K_1^M \subset J$. Since $K_*^M/(K_1^M) \cong \mathbb{Z}$, J is determined by its characteristic. \square

For the homogeneous prime ideals in $K_*^M/(2)$, one could simply rely on the now resolved Milnor conjecture and move the characterization of ideals in L^* through the quotient by I , but the story here is of independent interest.

Theorem 3.10. *Every element of $\text{Spec}^h(K_*^M/(2))$ is either (K_1^M) , or $([P_\alpha])$ for some order α .*

Proof. Suppose $J \in \text{Spec}^h(K_*^M/(2))$ and $J \neq (K_1^M)$. We have that, for all $a \in F^\times$

$$[a, -a] = 0$$

so, either $[a]$ or $[-a] = [-1] + [a]$ is in J . Thus, for some function $\alpha \in \{\pm 1\}^{F^\times}$,

$$[a] \equiv [\alpha(a)] \pmod{J}.$$

Since J is assumed to not contain K_1^M , it follows that $[-1] \notin J$, and α is onto. We will show that α is an ordering. First, α is multiplicative:

$$[ab] = [a] + [b] \equiv [\alpha(a)] + [\alpha(b)] = [\alpha(a)\alpha(b)] \pmod{J}.$$

And, α is additive. Suppose $\alpha(a) = \alpha(b) = 1$:

$$[a + b][-ba^{-1}] = [1 + ba^{-1}][-ba^{-1}] + [a][-ba^{-1}] \equiv 0 \pmod{J}.$$

Since $[-1] \notin J$, $[a + b] \in J$, and $\alpha(a + b) = 1$. □

Efrat has studied ordering and prime ideals in K_*^M using a kind of quotient construction K_*^M/G where G is a subgroup of F^\times [2]. It is worth noting that, in positive degrees, passing to the quotient by $([P_\alpha])$ is equivalent to passing to K_*^M/S , where S is the subgroup of α -positive elements. (In degree 0, Efrat's construction is \mathbb{Z} , where our quotient is $\mathbb{Z}/2\mathbb{Z}$.)

The topology of the minimum spectrum is reducible to the Harrison topology.

Theorem 3.11. $\text{MinSpec}^h(K_*^M)$ is homeomorphic to $X_F \amalg \{(K_1^M)\}$.

Proof. Let $\sigma_2 : X_F \rightarrow \text{MinSpec}^h(K_*^M)$ be the obvious injection, $\sigma_2(\alpha) = ([P_\alpha], 2)$. From previous computation, $\text{MinSpec}^h(K_*^M) = \sigma_2(X_F) \sqcup \{(K_1^M)\}$. We have that $D([-1]) = \sigma_2(X_F)$ and $D(2) = \{(K_1^M)\}$. So, it suffices to show that σ_2 is continuous and open.

First, σ_2 is open:

$$\begin{aligned} \sigma_2(H(a)) &= \{\sigma_2(\alpha) : \alpha(a) = 1\} \\ &= \{\sigma_2(\alpha) : \alpha(-a) = -1\} \\ &= \{\sigma_2(\alpha) : [-a] \notin \sigma_2(\alpha)\} \\ &= D([-a]) \end{aligned}$$

And, σ_2 is continuous. Consider some homogeneous $q = \sum_j [a_{j1}, \dots, a_{jn}]$. For any $\alpha \in X_F$, we have that

$$q \equiv m_\alpha [-1]^n \pmod{\sigma_2(\alpha)}$$

where $m_\alpha = |\{j : \forall i (\alpha(a_{ji}) = -1)\}|$. We also have

$$\text{sgn}_\alpha \left(\sum_j \langle -a_{j1}, \dots, -a_{jn} \rangle \right) = m_\alpha 2^n,$$

where $\langle\langle -a_{j1}, \dots, -a_{jn} \rangle\rangle = \prod_{1 \leq i \leq n} \langle 1, -a_{ji} \rangle$ is the Pfister form⁶ associated to $-a_{j1}, \dots, -a_{jn}$. Putting these together, we get

$$\begin{aligned} \sigma_2^{-1}(D(q)) &= \{\alpha : 2 \nmid m_\alpha\} \\ &= \operatorname{sgn} \left(\sum_j \langle\langle -a_{j1}, \dots, -a_{jn} \rangle\rangle \right)^{-1} (2^n \mathbb{Z} \setminus 2^{n+1} \mathbb{Z}) \end{aligned}$$

which is open. \square

3.3. Determination of $\operatorname{Spec}^h(K_*^{MW})$. All that remains is to piece the spectra of the quotients together.

Theorem 3.12. *Let $J \in \operatorname{Spec}^h(K_*^{MW})$. Then, J is exactly one of the following:*

- (1) $([P_\alpha], h, p)$ for some $\alpha \in X_F, p$ an odd prime,
- (2) $([P_\alpha], h)$ for some $\alpha \in X_F$,
- (3) $([P_\alpha], 2, \eta)$ for some $\alpha \in X_F$,
- (4) $([F^\times], p, \eta)$ for some p an odd prime,
- (5) $([F^\times], \eta)$,
- (6) $([F^\times], 2)$, or
- (7) $([F^\times], 2, \eta)$.

Proof. Note that these ideals are exactly those which arise by pulling the spectra of the quotients back along the quotient maps. An ideal in $K_*^{MW}/(\eta)$ contains h if and only if it contains 2, so if it is one of

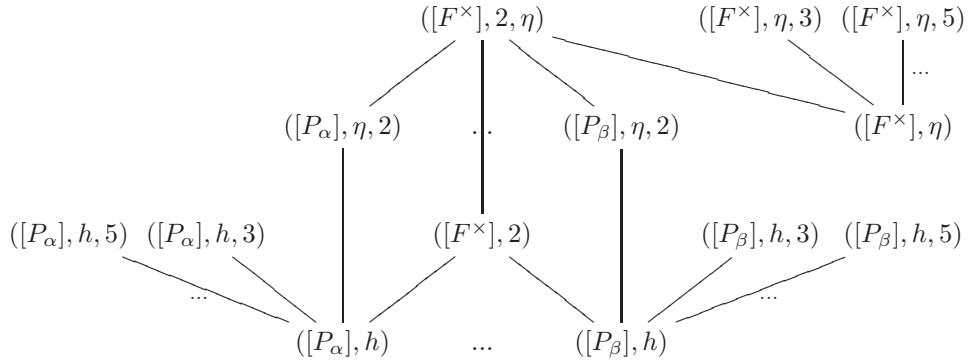
$$([F^\times], 2) \text{ or } ([P_\alpha], 2) \text{ for some } \alpha \in X_F.$$

These pull back to $([F^\times], 2, \eta)$ and $([P_\alpha], 2, \eta)$, which descend in $K_*^{MW}/(h)$ to

$$([F^\times], \eta) \text{ or } ([P_\alpha], 2, \eta).$$

\square

The Hasse diagram for the inclusion poset of these seven types of ideals is given below.



All that is left to prove Theorem 2.4 is to note that the inclusion maps

$$\operatorname{Spec}^h(K_*^{MW}/(h)) \rightarrow \operatorname{Spec}^h(K_*^{MW})$$

⁶According to at least one convention for Pfister forms.

and

$$\mathrm{Spec}^h(K_*^{MW}/(\eta)) \rightarrow \mathrm{Spec}^h(K_*^{MW})$$

are homeomorphisms onto their images. Topologically, then, the spectrum of K_*^{MW} is an X_F -like spray of copies of the rational primes, with the prime 2 tripled, all glued together at two of the copies of 2, along with another copy of the rational primes at the “top” of the diagram, with the closure of a set given by its upward closure in the inclusion poset pictured above.

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